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BOUNDARY EXTRAPOLATION IN HEAT TRANSFER CALCULATIONS

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NOMENCLATURE

B_i,	Biot number, hL/k or hr_0/k;
F_0,	Fourier number, $\alpha t/L^2$ or $\alpha t/r_0^2$;
h,	heat transfer coefficient;
J_0, J_1,	Bessel functions;
k,	thermal conductivity
L,	half-thickness of slab;
r_0,	radius of sphere or cylinder;
r, x,	coordinates;
R, X,	dimensionless coordinates r/r_0 or x/L;
t,	time;
T,	temperature.

Greek symbols

α,	thermal diffusivity;
β,	eigenvalues;
ε,	extrapolated distance;
θ,	dimensionless temperature, $(T - T_\infty)/(T_0 - T_\infty)$.

Subscripts

0,	initial condition;
∞,	ambient condition;
n,	order of eigenvalues;
ex,	exact;
ext,	extrapolated.

1. INTRODUCTION

THE TEMPERATURE distribution in heat transfer problems satisfies a differential equation. The differential equation together with initial and boundary conditions gives a solution for the temperature distribution at any point in the medium. The convective boundary condition, also known as the boundary condition of the third kind, is of prime importance in heat transfer calculations. The temperature distribution will be more readily obtained if the convective boundary condition is replaced with a known temperature at a new boundary. This is done by extrapolating the boundary and specifying the temperature on the new boundary.

In this paper a model employing a linear boundary extrapolation (from now on referred to as LBE model) is proposed which readily provides the information on the extrapolated boundary. The use of the linear extrapolated boundary condition has been very successful in neutron diffusion problems [1]. More sophisticated models using neutron transport theory have also been introduced [2, 3].

The transient temperature distribution, in the problem of heat transfer by conduction can be solved by separation of variables in many coordinates of interest. In the case of convective boundary condition the eigenvalues are solutions of a transcendental equation which involves the eigenfunctions as well as its derivatives. These eigenvalues have been found for some coordinates of practical importance, but have not been listed for other coordinate systems [4]. If the temperature is known on the extrapolated boundary, the eigenvalues are either known or can be calculated easier than the case of convective boundary condition.

The purpose of this paper is to find the range of applicability of the proposed model by comparing the exact solutions with those obtained from LBE model for different geometries.

2. THEORY

Consider a convective boundary condition at a planar boundary $x = L$. It is easy to show that for planar boundaries the linear extrapolation distance is $\varepsilon = k/h$. Therefore for the LBE model the boundary condition is

$$T = T_\infty, \text{ the ambient temperature when } x = L + k/h. \qquad (1)$$

It should be pointed out that the temperature distribution using the extrapolated boundary is meaningful in the domain of the problem and it is not valid outside the actual boundary of the problem. Furthermore, the treatment given here is valid for planar boundaries, and it is approximately correct for concave (non-reentrant) boundaries [2]. In this paper the temperature distributions of infinite slab, infinite cylinder and sphere for which exact solutions can be found [5] are considered. Since the boundary value problems involved are mathematically simple, only the results of the LBE model for the three geometries are given below. The generalization to the three dimensional problems is straightforward.

(a) *Infinite slab of thickness* 2L

The dimensionless temperature distribution is given by

$$\frac{\theta}{\theta_0} = \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n-1} \cos\left[\frac{2n-1}{2}\pi\left(\frac{B_i}{1+B_i}\right)X\right] \exp -\left(\frac{2n-1}{2}\pi\frac{B_i}{1+B_i}\right)^2 F_0 \qquad (2)$$

where the dimensionless coordinate, Fourier number and Biot number are respectively

$$X = x/L, F_0 = \alpha t/L^2, \text{and } B_i = hL/k.$$

(b) *Infinite cylinder of radius* r_0

$$\frac{\theta}{\theta_0} = \sum_{n=1}^{\infty} \frac{2}{\beta_n J_1(\beta_n)} J_0\left(\beta_n R \frac{B_i}{1+B_i}\right) \exp -\left(\frac{\beta_n B_i}{1+B_i}\right)^2 F_0 \quad (3)$$

where β_n's are the zeroes of Bessel function J_0, $R = r/r_0$, $F_0 = \alpha t/r_0^2$, and $B_i = hr_0/k$.

(c) *Sphere of radius* r_0

$$\frac{\theta}{\theta_0} = \sum_{n=1}^{\infty} \frac{(2)(-1)^{n-1}}{n\pi R}\left(\frac{1+B_i}{B_i}\right) \sin\left(n\pi\frac{B_i R}{1+B_i}\right) \exp -\left(n\pi\frac{B_i}{1+B_i}\right)^2 F_0 \qquad (4)$$

where $B_i = hr_0/k$ and $F_0 = \alpha t/r_0^2$.

Note that the temperature distribution using the LBE model does not require solution of a transcendental equation for evaluation of eigenvalues.

3. RESULTS AND DISCUSSION

To investigate the applicability of the LBE model the exact solutions for slab, cylinder, and sphere [5] are compared with solutions resulting from the LBE model. For the comparison the per cent error is defined as

$$\text{Per cent error} = \left|\frac{\theta_{ex} - \theta_{ext}}{\theta_{ex}}\right| \times 100. \qquad (5)$$

For infinite Biot number the extrapolated boundary coincides with the actual boundary; the eigenvalues for the two cases will be equal and the error is zero. The solid curves in Fig. 1 show the percent error for temperature at the midplane of slab. From Fig. 1 it is seen that the error decreases as the Biot number increases and hence the extrapolation distance decreases. The error also decreases for small values of the Fourier number. The error becomes negligible when the Biot number gets to be large.

It may seem that for large Biot numbers we can do without the boundary extrapolation by assuming the temperature

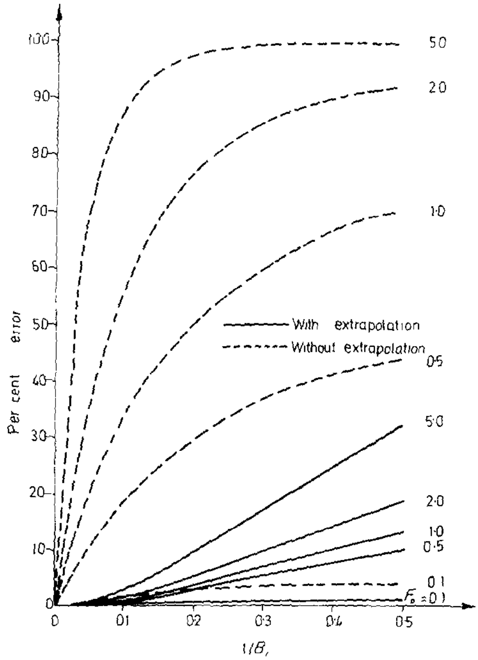

FIG. 1. Per cent error for the midplane slab temperature, as a function of the Biot number for different Fourier numbers.

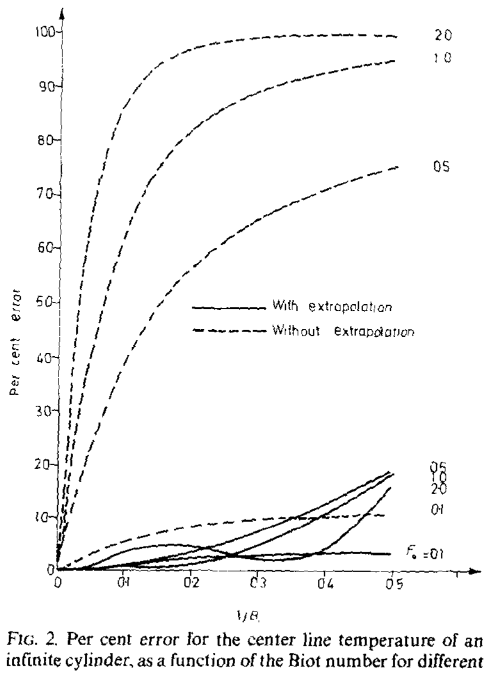

FIG. 2. Per cent error for the center line temperature of an infinite cylinder, as a function of the Biot number for different Fourier numbers.

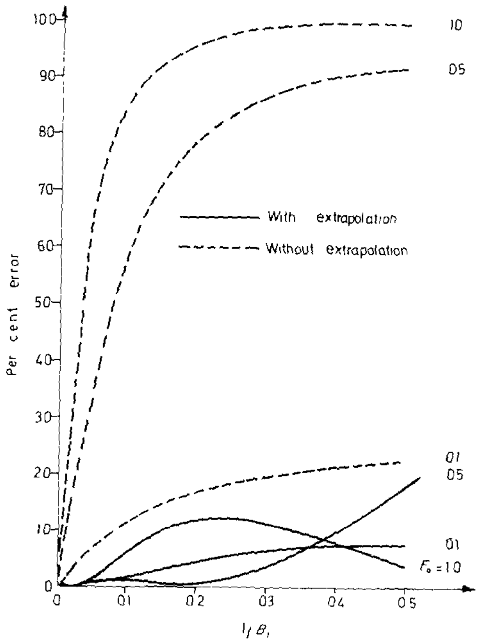

FIG. 3. Per cent error for the temperature at center of a sphere, as a function of the Biot number for different Fourier numbers.

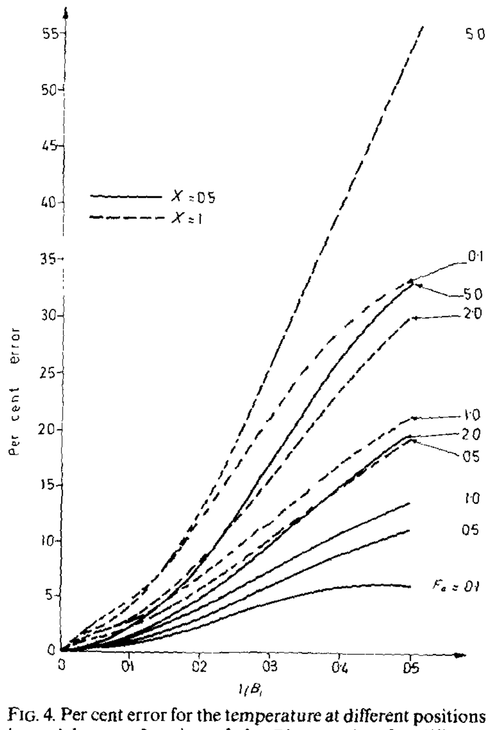

FIG. 4. Per cent error for the temperature at different positions in a slab, as a function of the Biot number for different Fourier numbers.

on the actual boundary to be equal to the ambient temperature. The dashed curves in Fig. 1 represent the percent error if the actual boundary is used instead of the linear extrapolated boundary. It is informative to note that the error jumps to about 70 per cent for $F_0 = 5$ and $B_i = 20$ if the LBE model is not used compared to an error of 1 per cent if the LBE model is used. This clearly shows the importance of the linear boundary extrapolation. The difference in error using the actual boundary and the LBE model becomes more pronounced for large Fourier numbers.

Figures 2 and 3 show the results for the cylinder and sphere. The regular behaviour of error curves of slab in Fig. 1 is seen to be absent in the cylinder sphere examples. The irregular behaviour of error curves for cylinder and sphere is due to the fact that for these geometries the extrapolated distance is a function of the curvature [2] and its value is not simply k/h as obtained for planar boundaries. We

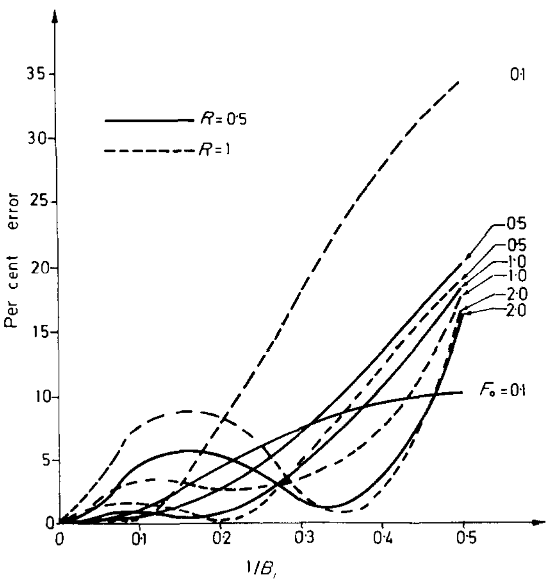

FIG. 5. Per cent error for the temperature at different positions in an infinite cylinder, as a function of the Biot number for different Fourier numbers.

observed that the error for cylinder and sphere can be in most cases reduced if a modified extrapolation distance greater than k/h is used. The error with the modified boundary is very sensitive to value of the extrapolation distance. For large Biot numbers the error for cylinder and sphere (cf. Figs. 2 and 3) is small even if k/h is used as the extrapolated distance.

Figure 4 shows the error for slab using the LBE model for different Biot numbers with the Fourier number being used as a parameter. The solid and the dashed curves represent the error for $x = 0{\cdot}5$ and $x = 1$ respectively. In general as the distance from the slab center increases so does the error. This behaviour is due to the fact that as one gets closer to the center of the slab the effect of a change in the boundary gets smaller.

Figures 5 and 6 show the results for the cylinder and sphere respectively. The solid curves represent the error for $R = 0{\cdot}5$ and the dashed curves show the error for the surface temperature, i.e. $R = 1$. The irregular behaviour of the error curves is again due to the use of a simple planar extrapolation distance k/h for cylinder and sphere.

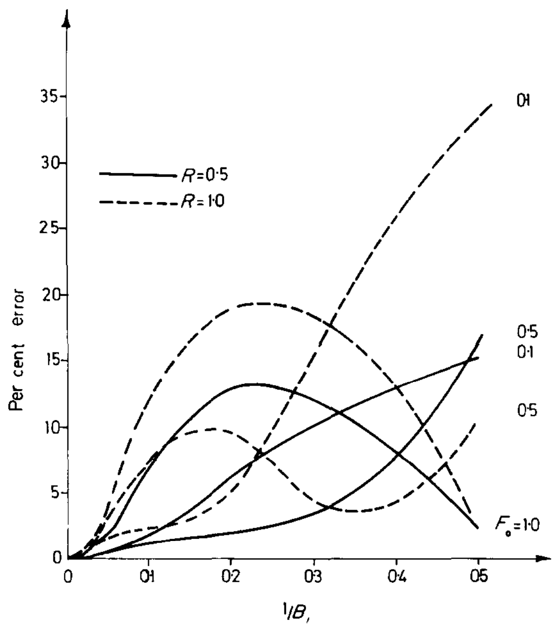

FIG. 6. Per cent error for the temperature at different positions in a sphere, as a function of the Biot number for different Fourier numbers.

Several extrapolation distances other than k/h were used for the cylinder and sphere in an attempt to reduce the magnitude of the error. Although the results were favourable, no general conclusion could be reached from them. If the first eigenvalue for an exact solution is known, the extrapolation distance can be modified so that the same temperature is obtained for a given Fourier number. Second and higher eigenvalues will be obtained using the modified extrapolation based on the value of the first exact eigenvalue. The temperature distribution using these eigenvalues is expected to give a better result than using the extrapolation distance k/h. This technique has the advantage of giving the correct result for large Fourier numbers, since for large Fourier numbers the first eigenvalue plays the dominant role in the temperature distribution.

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INDIRECT THERMAL SENSING IN COMPOSITE MEDIA

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NOMENCLATURE

$A_{ij}, A_{in}, B_{ij}, B_{in}$, constants;
C_{pi}, specific heat at constant pressure in the ith section;
D_i^2, thermal diffusivity in the ith section;
E_{in}, constant;
i, j, k, integers;
J_0, zero order Bessel function of the first kind;
K_i, thermal conductivity in the ith section;
$K_{jA}(x, t), K_{jB}(x, t), K_{jC}(x, t)$, functions of space and time, defined by equations (17)–(19);
M_{in}, N_{in}, derived eigenfunction terms (dimensionless);
Q_i, distributed source in the ith section;
t, time;
$T_i(x, t)$, temperature in the ith section;
$X_{in}(x)$, eigenfunction (dimensionless);
x, spatial coordinate;
Y_0, zero order Bessel function of the second kind;
γ_n, eigenvalue;
ρ_i, density in the ith section.

INTRODUCTION

The problems associated with obtaining direct experimental data in extreme environments and when space for data probes is not available has led logically to a program of indirect experimental measurements. The use of proper analytical techniques, along with indirect experimental data, allows the determination of the desired information at locations inaccessible directly by experimental probes.

In this type of problem one seeks the transient boundary conditions given the initial and some time-dependent conditions in the interior of the media. This is called an inverse problem or an interior value problem in contrast to a boundary value problem.

At present, the only technique for the solution of the inverse problem in composite media is a numerical method proposed by Beck [1]. Mulholland and San Martin [2] used the results of a known exact solution to obtain the internal and external temperature history of a composite. The objective of this paper is to present an analytical method which builds on the ideas presented in [2] for treating such problems in composite materials composed of k solidly joined plates, cylinders or spheres.

STATEMENT OF PROBLEM

The heat conduction equation for the ith section of k solidly joined plates, cylinders or spheres is